Algebraic Approach to the Fixed Point Structure of the Quantum Mechanical Dirac-Coulomb System

Nivedita Nag and Rajkumar Roychoudhury
Physics and Applied Mathematics Unit Indian Statistical Institute, Calcutta-700035

Z. Naturforsch. **50a,** 995–997 (1995); received June 30, 1995

It is shown that a non-perturbative β like function can be obtained for a Dirac-Coulomb system with both vector and scalar couplings using the properties of $O(2,1)$ algebra and the tilting operator mechanism.

Ten years ago Leung et al. [1] estimated the non-perturbative β function in QED [2–5] in the super critical phase as

$$\beta_c(\alpha) = -\frac{2\alpha_c^*}{\theta}\left(\frac{\alpha}{\alpha_c^*}-1\right)^{3/2}, \tag{1}$$

where $\alpha_c^* = \frac{1}{3}\pi$, $0<\theta\leq\pi$ for the ground state. Later Papp [6] applied the quantum mechanical $1/N$ method to obtain the β function in the quantum mechanical Dirac-Coulomb system. However the $1/N$ description of the Dirac-Coulomb system, though non-perturbative in the sense that it is valid both for small and large coupling constants, is perturbative in the parameter $1/N$. We here show that a more elegant and straight forward technique, viz. the $SO(2,1)$ algebraic method can be applied to the quantum mechanical Dirac-Coulomb system to obtain β like functions. We define the β-function

$$\beta(\alpha) = -x\frac{\partial\alpha}{\partial x_0},$$

where x_0 is related to the tilting operator θ, which will be defined below.

Here we use the $O(2,1)$ [7–9] formalism to obtain exact results for a potential $V(r)$ with $V_v=-\alpha/r$ and $V_s=-a/r$, of V. We also use the scaling variational approach to show that, though this approach is supposed to give only approximate results, it yields exact eigenvalues in this particular case. The Dirac equation for a potential with vector and scalar parts equal to $-\alpha/r$ and $-a/r$, respectively, is given by

$$H\Psi = E\Psi, \tag{2a}$$

Reprint requests to Prof. R. Roychoudhury.

where

$$H = \boldsymbol{\alpha} - \boldsymbol{p} + \beta\left(m-\frac{a}{r}\right) - \frac{\alpha}{r} \tag{2b}$$

(the units are such that $\hbar=c=1$).

The generators of $O(2,1)$ algebra appropriate for the Hamiltonian (2) are given by (these are the generalized versions of the operator given in [7])

$$\Gamma_0 = \frac{1}{2}\left[r p^2 + r + \frac{1}{2}\left\{-\alpha^2+a^2-i(\alpha\boldsymbol{\alpha r}-\beta a\boldsymbol{\alpha r})\right\}\right], \tag{3a}$$

$$\Gamma_4 = \frac{1}{2}\left[r p^2 - r + \frac{1}{r}\left\{-\alpha^2+a^2-i(\alpha\boldsymbol{\alpha r}-\beta a\boldsymbol{\alpha r})\right\}\right], \tag{3b}$$

$$T = \boldsymbol{r p} - i, \tag{3c}$$

where $\boldsymbol{p} = \frac{\hbar}{i}\boldsymbol{\nabla}$.

The generators satisfy the commutation relations of the Lie algebra of the group $O(2,1)$, which are (using notations of [7]) given by

$$[\Gamma_0,\Gamma_4] = iT,\quad [\Gamma_4,T] = -i\Gamma_0,\quad [T,\Gamma_0] = i\Gamma_4. \tag{4}$$

The Casimir operator is given by

$$Q^2 = \Gamma_0^2-\Gamma_4^2-T^2 = J^2-\alpha^2+a^2-i(\alpha-\beta a)\boldsymbol{\alpha r} = n'(n'+1), \tag{5}$$

n' denoting the eigenvalue of Q. Q^2 can be written as

$$Q^2 = \Gamma^2 - \Gamma, \tag{6}$$

where

$$\Gamma = \boldsymbol{\sigma J} + i(\alpha-\beta a)\boldsymbol{\alpha} - \boldsymbol{r} + 1. \tag{7}$$

Γ has the eigenvalues

$$\gamma = \pm\left[\left(j+\frac{1}{2}\right)^2 - \alpha^2 + a^2\right]^{1/2}, \tag{8}$$

0932-0784 / 95 / 1100-0995 $ 06.00 © – Verlag der Zeitschrift für Naturforschung, D-72027 Tübingen

where j is the total angular momentum. From (5) we get

$$n' = -\gamma \quad \text{or} \quad \gamma - 1. \tag{9}$$

If we diagonalize Γ_0 simultaneously with Q^2, then the eigenvalues of Γ_0 will be given by

$$\bar{n} = \gamma + n, \tag{10}$$

where $n = 0, 1, 2, \ldots$.

With the condition that if $j = 1 - ½$, n will not take the value 0. From the Hamiltonian (2) we define the operator

$$\bar{\theta} = \left[r p^2 - (E^2 - m^2) r - 2\alpha E - 2am + \frac{1}{r}\left\{ -\alpha^2 + a^2 - i(\alpha - \beta a)\boldsymbol{\alpha}\boldsymbol{r} \right\} \right], \tag{11}$$

which is appropriate for the second order Dirac equation obtained by applying $\left(H + \frac{\alpha}{r}\right)$ on the equation

$$\left(H + \frac{\alpha}{r}\right)\Psi = \left(E + \frac{\alpha}{r}\right)\Psi. \tag{12}$$

Using (11), the second order Dirac equation can be expressed as

$$\bar{\theta}\Psi = 0. \tag{13}$$

In terms of the generators of $O(2,1)$ algebra

$$\bar{\theta} = \Gamma_0 + \Gamma_4 - (E^2 - m^2)(\Gamma_0 - \Gamma_4) - 2\alpha E - 2am. \tag{14}$$

We now use the tilting operator formalism by writing

$$\Psi = e^{i\theta T}\Psi. \tag{15}$$

The generators are transformed as [7]

$$\begin{aligned} \Gamma_0 &\to \Gamma_0 \cosh\theta + \Gamma_4 \sinh\theta, \\ \Gamma_4 &\to \Gamma_4 \cosh\theta + \Gamma_0 \sinh\theta. \end{aligned} \tag{16}$$

Equation (13) then changes to

$$\{\Gamma_4[e^\theta + (E^2 - m^2)e^{-\theta}] + \Gamma_0[e^\theta - (E^2 - m^2)e^{-\theta}] - 2\alpha E - 2am\}\Psi = 0. \tag{17}$$

Now we choose θ in such a manner that the coefficient of Γ_4 vanishes. Alternatively we can choose θ to minimize E. For non solvable cases only the second method is feasible. We discuss this in the following.

1. Diagonalization Method

Choose θ such that

$$e^{2\theta} = m^2 - E^2. \tag{18}$$

Then from (17), using $\Gamma_0\Psi = \bar{n}\Psi = (\gamma + n)\Psi$, we have

$$2\bar{n}\sqrt{m^2 - E^2} - 2\alpha E - 2am = 0. \tag{19}$$

Solving for E, we get

$$E = \frac{m}{(\gamma + n)^2 + \alpha^2} \cdot \left[(\gamma + n)\sqrt{(\gamma + n)^2 - a^2 + \alpha^2} - a\alpha \right], \tag{20}$$

which agrees completely with the result obtained by Tutik [10] if one takes $\gamma = 3$.

2. Scaling Variational Approach

From (17), taking only the diagonal part, we get

$$\bar{n}e^\theta - (E^2 - m^2)\bar{n}e^{-\theta} - 2\alpha E - 2am = 0. \tag{21}$$

We choose θ such that the energy is minimum, i.e.

$$\frac{\partial E}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 E}{\partial \theta^2} > 0.$$

Now $\frac{\partial E}{\partial \theta} = 0$ gives

$$\bar{n}e^\theta + (E^2 - m^2)\bar{n}e^{-\theta} = 0, \tag{22}$$

which means

$$e^{2\theta} = m^2 - E^2. \tag{23}$$

Equation (23) is identical with equation (18), and hence the scaling variational method also reproduces the exact result given by (20). Now, from (3a) and (3b)

$$\frac{1}{r} = \Gamma_0 - \Gamma_4. \tag{24}$$

Using the transformation (16) and taking the diagonal term, we get

$$r_{\text{diag}} = \frac{e^{-\theta}}{\bar{n}}. \tag{25}$$

We define x_0 as

$$x_0 = r_{\text{diag}} = \frac{e^{-\theta}}{\bar{n}}. \tag{26}$$

Using (23), (18), and (26) it can easily be seen that

$$x_0 = \frac{(1+a^2)}{m\bar{n}(a\bar{n}+\alpha)} . \tag{27}$$

In a normalization scheme, where a is independent of x_0,

$$\beta = -x_0 \frac{\partial \alpha}{\partial x_0} = \frac{\bar{n}^2}{(\bar{n}^2+\alpha^2)} (\alpha + a\bar{n}) , \tag{28}$$

where $\bar{n} = \sqrt{1-\alpha^2+a^2}$ for the ground state.

The result (28) is valid for the ground state. The value of β obtained from (28) is identical with that obtained by Papp [6] if one takes $N=3$. It is to be noted that x_0 is the location of the minimum for the ground state energy. Since our result is obtained for $N=3$, the other conclusions regarding the β function obtained [6] will only hold good if one takes $N=3$. For the special case $\alpha = a$, we find

$$\beta = \alpha \frac{1+\alpha^2}{1-\alpha^2} .$$

This result is also identical with the result obtained in [6] for $N=3$.

To conclude, the algebraic method can yield the quantum mechanical β function for the Dirac-Coulomb system in an elegant and straight forward way.

[1] C. N. Leung, S. T. Love, and W. A. Bardeen, Nucl. Phys. **B 273,** 649 (1969).
[2] V. A. Miransky, Nuovo Cim., A **90,** 149 (1985).
[3] J. B. Kogut, E. Dogotte, and A. Kocic, Phys. Rev. Lett. **61,** 2416 (1988).
[4] J. Bartholomew, S. H. Shenker, J. Sloan, J. Kogut, M. Stome, H. W. Wyld, J. Shingmetsu, and D. K. Sinclair, Nucl. Phys. B **230,** 222 (1989).
[5] P. I. Fomin, U. P. Gusynin, V. A. Miransky, and Yu. A. Sitenko, Rev. Nuovo Cim. **6,** 5 (1983).
[6] E. Papp, Phys. Lett B **288,** 148 (1992).
[7] A. O. Barut and G. L. Bornzin, J. Math. Phys. **12,** 841 (1971).
[8] A. O. Barut and J. Nagel, J. Phys. A **10,** 1243 (1977).
[9] A. O. Barut, Dynamical groups and generalized symmetries in quantum theory (Univ. of Canterbury Press, Christchurch, New Zealand 1971).
[10] R. S. Tutik, J. Phys. A Math. Gen. **25,** L413 (1992).